P-adic Coding and Computations

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Smart City Science

- „real-time systems for public good connecting public sector with industrial sector“

Aspects
- Standards
- Systems
- Governance balkanization
  - Gaia-X, architecture, Geo-spatial IOT, ...
- Algorithms/Methods
  - Distributed, Tractable, Efficient, Adequate, Ressource Aware
- Ethics
  - Privacy, Explainability, Participation and Knowledge Transfer
Smart City Science Topics

N.N.

Xeniya Gusseva (BSc)
HiWi (SFB-B4)

Fried Kullman (MSc)
Prediction of Machine State after Fault Injection

Karen Toben (MSc)
p-adic Neural Networks

Lukas Schneider (BSc)
Factorizing Distributions by Conditional Sum Product Networks

Timon Sachweh (MSc)
Differential Privacy for Learning from Label Proportions

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Prediction of traffic flow

Models and their Assumptions:

• Spatio-Temporal Random Fields
  – Markov Assumption
  – Tobler‘s Law
  – Daily Routines
• Distributed Prediction
  – Tobler‘s Law

• Gaussian Process Regression
  – Central limit theorem
  – Tobler‘s Law
• Graph CNN
  – Markov Assumption
  – Tobler‘s Law
  – Network topology
• Conditional Sum-Product Networks
  – Discrete distribution
  – Markov Assumption
We made implicit Assumptions

• Since Newton/Leibniz Spatio-Temporal phenomena modelled by ODE or PDE on $\mathbb{R}$ (Euclidean or Minkowski coordinates)

• Models in $\mathbb{R}$ map to $\mathbb{R}$

• Veronese & Hilbert stated in Euclidean geometry holds Archimedean Axiom

\[ b > a \implies \exists m : b < m \cdot a \]
Problem with these Assumptions

• But, our observations can not be infinitesimal and are bound to topology of traffic network

→ Abandon Archimedean Axiom at very small scale (e.g. innercity junctions with low temporal granularity)

• How?
Geometry and Number Systems

- Coordinates describe Geometric picture
  \[
  \mathbb{R} \quad \text{Euclidean geometry}
  \]
  \[
  ? \quad \text{Non-Euclidean geometry}
  \]

- In computations or for measurements we use \( \mathbb{Q} \)
  \rightarrow \text{field (} \mathbb{Q}, | \cdot | \text{)}

  \[
  |x| = 0 \iff x = 0
  \]
  \[
  |xy| = |x||y|
  \]
  \[
  |x + y| \leq |x| + |y|
  \]
TwoPossibilitiesfor$|\cdot| $ [Ostrowski 16]

$|\cdot| : \mathbb{Q} \rightarrow \mathbb{Q}_+$

$|x| = \begin{cases} 
  x & \text{if } x \geq 0, \\
  -x & \text{if } x < 0.
\end{cases}$

- Completion of $(\mathbb{Q}, |\cdot|)$ leads to $(\mathbb{R}, |\cdot|)$
The Geometry of $\mathbb{Q}_p$

- $\mathbb{R}$ and $\mathbb{Q}_p$ are metric spaces
- $(\mathbb{R},|\cdot|)$ and $(\mathbb{Q}_p,|\cdot|_p)$ are very different!
- In a metric space $(X,d)$ the open balls are sets
  \[ U_r = \{ x \in X : d(a,x) < r \} \]
  \[ \text{in } (\mathbb{R},|\cdot|) \quad U_r = \{ x \in \mathbb{R} : |a-x| < r \} = (a-r,a+r) \]
  \[ \text{in } (\mathbb{Q}_p,|\cdot|_p) \quad U_r = \{ x \in \mathbb{Q}_p : |a-x|_p < r \} \]
Applications of $p$-adic Models

- Turbulence
- Dynamic Systems
- Cryptography
- Economy
- Chaotic Fractal Behavior

- Quantum Mechanics
- Hierarchical Models
- Neuro Cognition
Classification of $p$-adic vectors

Q_p^N = \mathbb{Q} \times \mathbb{Q} \times \ldots \times \mathbb{Q}

\|x\|_p = \max |x_j|_p \text{ with: } x = (x_0, \ldots, x_{N-1}) \in Q_p^N

- Example \hspace{1cm} x = (x_1, x_2)^T \in Q_7^2

| |x_1 + x_2|_7 < 0.5 \hspace{1cm} \|x\|_7 < 0.5
Classification of p-adic vectors

- Sphere

\[ \| x \cdot c^T \|_2 - b = 0 \]

selects one of these colors
Classification by digits of p-adic extension

\[ x = (x_1, x_2)^T \in \mathbb{Q}_7^2 \]

\[ [x_1^2 + x_2^2]_7 \equiv 2 \ldots \]

\[ [x_1^2 + x_2^2]_7 \equiv 23 \ldots \]
Negative probabilities

• In Kolmogorov's probability framework probabilities of events must be **positive real numbers** [Kolmogorov 31]

• Here, we consider ensemble frequency of an ensemble of balls [Mises 19]
  
  – Consider, countable number of colors \( C \)
  
  – We observe an ensemble \( S \) of colored balls with the 
    \[ \text{#num of balls per color } k = 2^k \]
Negative probabilities

• Here, we consider ensemble frequency of an ensemble of balls
  – Consider, countable number of colors
  – We observe an ensemble $S$ of colored balls with the number of balls per color $k = 2^k$

• 'Volume' $N = |S|$ of $S$ is

$$N = \sum_{k=0}^{\infty} n_k = \sum_{k=0}^{\infty} 2^k$$
Negative probabilities

- This sum diverges in $\mathbb{R}$

$$N = \sum_{k=0}^{\infty} n_k = \sum_{k=0}^{\infty} 2^k$$

**but** converges in $\mathbb{Q}_2$

$$N = \sum_{k=0}^{\infty} 2^k = \frac{1}{1 - 2} = -1$$
Interesting to explore

• Models in $\mathbb{Q}_p$
• Model for (chaotic) cellular automaton (e.g. sandpile avalanche process)
• Negative & complex probabilities
  Description of Quasiprobabilities (e.g. Wigner distribution)
• Utilization of $p$-adic Algorithms in ressource constraint devices e.g. matrix inversion by [Dixon 82] ring inversion
  [Koç 17] random graphs [Hua & Hovestadt 21]
Literature